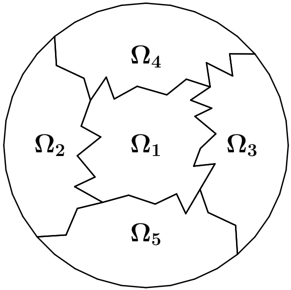
Generalized Optimised Schwarz Method for arbitrary non-overlapping subdomain partitions

X.Claeys[†] & E.Parolin[*]

[†] Laboratoire Jacques-Louis Lions, Sorbonne Université
INRIA Paris, équipe Alpines
[*] POems, UMR CNRS/ENSTA/INRIA, ENSTA ParisTech

Scattering in heterogeneous medium

wave number : $\kappa : \mathbb{R}^d \to \mathbb{C}$ bounded
$\Re e\{\kappa(\boldsymbol{x})\} \geq 0, \Im m\{\kappa(\boldsymbol{x})\} \geq 0, \kappa(\boldsymbol{x}) \neq 0$
source : $f \in \mathrm{L}^2(\Omega)$



Non-overlapping partition
$\Omega = \overline{\Omega}_1 \cup \cdots \cup \overline{\Omega}_{\mathrm{J}}$,
$\Gamma_j := \partial\Omega_j, \quad \Gamma'_j := \Gamma_j \setminus \partial\Omega$
Ω_j : Lipschitz, bounded

Helmholtz bvp
$-\Delta u - \kappa(\boldsymbol{x})^2 u = f$ in Ω,
$\partial_{\boldsymbol{n}} u - \imath\kappa\, u = 0$ on $\partial\Omega$.

$\Longleftrightarrow$

local sub-problems $j = 1 \ldots \mathrm{J}$
$-\Delta u - \kappa^2 u = f$ in Ω_j
$\partial_{\boldsymbol{n}} u - \imath\kappa\, u = 0$ on $\partial\Omega_j \cap \partial\Omega$.

+

transmission conditions
$\partial_{n_j} u|_{\Gamma_j}^{\mathrm{int}} = -\partial_{n_k} u|_{\Gamma_k}^{\mathrm{int}}$
$u|_{\Gamma_j}^{\mathrm{int}} = u|_{\Gamma_k}^{\mathrm{int}} \quad \forall j, k$

Cross points allowed

Optimized Schwarz Method (OSM) [Després, 1991]

Optimized Schwarz Method (OSM) is one of the most established DDM approaches for wave propagation. This is a substructuring method where transmission conditions are imposed through each interface by means of Robin traces involving impedance coefficients.

- operator valued impedance : [Collino, Ghanemi & Joly, 2000]
- second order TC : [Gander, Magoules & Nataf, 2002]
- DtN-like impedance : [Nataf, Rogier & de Sturler, 1995], [Antoine, Boudendir & Geuzaine, 2012], [Antoine, Bouajaj & Geuzaine, 2014]
- large literature : overview article [Gander & Zhang, 2019]

Cross point issue

Unappropriate treatment of cross-points may spoil convergence so care must be paid to this issue : [Gander & Kwok, 2013], [Gander & Santugini, 2016], [Després, Nicolopoulos & Thierry, 2020], [Modave, Antoine, Geuzaine & al, 2019 & 2020]. There is also a variant of FETI-DP "à la Després" [Farhat & al, 2005], [Bendali & Boubendir, 2006].

Outline

I Review of the Optimized Schwarz Method

II New manner to enforce transmission condition

III Numerical results

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Original OSM (Després)

Transmission conditions : with scalar $\Lambda > 0$,

$$\begin{array}{c} \partial_{n_j} u|_{\Gamma_j} = -\partial_{n_k} u|_{\Gamma_k} \\ u|_{\Gamma_j} = u|_{\Gamma_k} \\ \text{on } \Gamma_j \cap \Gamma_k \ \forall j,k \end{array} \iff \begin{array}{c} +\partial_{n_j} u|_{\Gamma_j} + \imath\Lambda u|_{\Gamma_j} = \\ -\partial_{n_k} u|_{\Gamma_k} + \imath\Lambda u|_{\Gamma_k} \\ \text{on } \Gamma_j \cap \Gamma_k \ \forall j,k \end{array} \iff \begin{array}{l} (\partial_{n_j} u|_{\Gamma_j'} - \imath\Lambda u|_{\Gamma_j'})_{j=1}^{\mathrm{J}} = \\ -\Pi_0((\partial_{n_k} u|_{\Gamma_k'} + \imath\Lambda u|_{\Gamma_k'})_{k=1}^{\mathrm{J}}) \end{array}$$

where the operator Π_0 swaps traces on both sides of each interfaces :
$(v_0, \ldots, v_{\mathrm{J}}) = \Pi_0(u_0, \ldots, u_{\mathrm{J}}) \iff v_j = u_k$ on $\Gamma_j \cap \Gamma_k$.

Local scattering operators :

$\mathrm{S}_0^{\Gamma_j}(\partial_{n_j}\psi|_{\Gamma_j'} - \imath\Lambda\psi|_{\Gamma_j'}) := \partial_{n_j}\psi|_{\Gamma_j'} + \imath\Lambda\psi|_{\Gamma_j'}$

for $\Delta\psi + \kappa^2\psi = 0$ in Ω_j

$\partial_{\boldsymbol{n}}\psi - \imath\kappa\psi = 0$ on $\partial\Omega_j \cap \partial\Omega$

Wave equations :

$(\partial_{n_j} u|_{\Gamma_j'} + \imath\Lambda u|_{\Gamma_j'})_{j=1}^{\mathrm{J}} =$
$\mathrm{S}_0((\partial_{n_k} u|_{\Gamma_k'} - \imath\Lambda u|_{\Gamma_k'})_{k=1}^{\mathrm{J}}) + g$

with $\mathrm{S}_0 := \mathrm{diag}_{j=1\ldots\mathrm{J}}(\mathrm{S}_0^{\Gamma_j})$.

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stems from the source term of the bvp

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on $\Gamma_j \cap \Gamma_k \; \forall j, k$ on $\Gamma_j \cap \Gamma_k \; \forall j, k$

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Optimized Schwarz

$$(\mathrm{Id} + \Pi_0\mathrm{S}_0)p = -\Pi_0(g)$$

with $p = (\partial_{n_j} u|_{\Gamma'_j} - \imath\Lambda u|_{\Gamma'_j})_{j=1}^{\mathrm{J}}$

The cross-point issue

$$p^{(n+1)} = (1-r)p^{(n)} - r\Pi_0 \mathrm{S}_0(p^{(n)}) - r\Pi_0(g)$$

Without cross points, geometric convergence can be obtained for appropriate (operator valued) impedance Λ. In practice, convergence is much slower with cross points i.e. at best algebraic $\|p - p^{(n)}\|_{\mathrm{L}^2} = O(n^{-\gamma})$. The root cause seems related to Π_0 not being continuous at cross-points in proper trace norms.

This so-called "cross point issue" also arises in the different context of multi-domain boundary integral formulations where Multi-Trace Formalism (MTF) [Claeys & Hiptmair, 2012] now offers a framework that accomodates cross-points, and that is clean as regards function spaces.

Idea : use the Multi-Trace Formalism to treat cross-points in OSM. We shall replace Π_0 by a non-local counterpart Π that remains continuous no matter the presence of cross points, following the idea first introduced in :

X.Claeys, "Quasi-local multi-trace boundary integral formulations", Numer. Methods Partial Differential Equations, 31(6) :2043–2062, 2015.

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Discrete function spaces

Triangulation conforming with Ω_j's, FE spaces $\mathrm{V}_h(\Omega_j) = \{\ \mathbb{P}_k$-Lagrange on $\Omega_j\}$.

Volume functions	**Tuples of traces** ($\Gamma_j := \partial\Omega_j$)
$\mathbb{V}_h(\Omega) := \mathrm{V}_h(\Omega_1) \times \cdots \times \mathrm{V}_h(\Omega_\mathrm{J})$	$\mathbb{V}_h(\Sigma) := \mathrm{V}_h(\Gamma_1) \times \cdots \times \mathrm{V}_h(\Gamma_\mathrm{J})$
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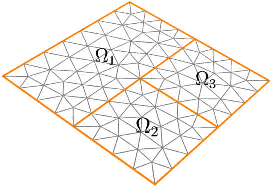

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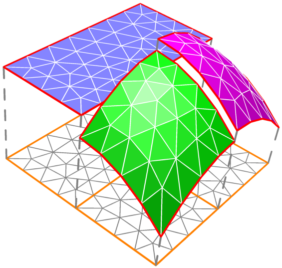
Piecewise H^1
Possible jumps through interfaces



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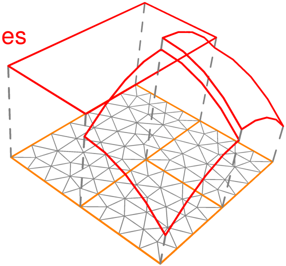
Multi-traces = tuples of traces at boundaries



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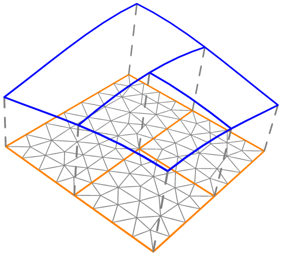
Single-traces = tuples of traces
that match at interfaces



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Impedance = scalar product on traces

$$t_h(\mathfrak{p}, \mathfrak{q}) = t_{\Gamma_1}(p_1, q_1) + \cdots + t_{\Gamma_{\mathrm{J}}}(p_{\mathrm{J}}, q_{\mathrm{J}})$$

$t_{\Gamma_j}(\cdot, \cdot) =$ **any** scalar product on $\mathrm{V}_h(\Gamma_j)$

Choices of impedance :

- surface mass matrix
- surface order 2 operator
- layer potential, DtN map
- Schur complement

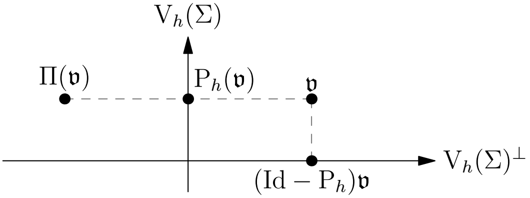
Matching at interfaces via orthogonal symmetry

The t_h-orthogonal projection onto the single-trace space $\mathrm{P}_h : \mathbb{V}_h(\Sigma) \to \mathrm{V}_h(\Sigma)$ can be applied by solving a (**DDM friendly !**) SPD problem

$$\mathfrak{p} = \mathrm{P}_h(\mathfrak{v}) \quad \iff \quad \begin{array}{l} \mathfrak{p} \in \mathrm{V}_h(\Sigma) \quad \text{and} \\ t_h(\mathfrak{p}, \mathfrak{w}) = t_h(\mathfrak{v}, \mathfrak{w}) \quad \forall \mathfrak{w} \in \mathrm{V}_h(\Sigma) \end{array}$$

Lemma : The t_h-orthogonal symmetry $\Pi := \mathrm{P}_h - (\mathrm{Id} - \mathrm{P}_h) = 2\mathrm{P}_h - \mathrm{Id}$ satisfies $\|\Pi(\mathfrak{q})\|_{t_h} = \|\mathfrak{q}\|_{t_h} \forall \mathfrak{q} \in \mathbb{V}_h(\Sigma)$ and, for any $\mathfrak{v}, \mathfrak{q} \in \mathbb{V}_h(\Sigma)$,

$$(\mathfrak{v}, \mathfrak{q}) \in \mathrm{V}_h(\Sigma) \times \mathrm{V}_h(\Sigma)^{\perp} \quad \iff \quad \mathfrak{q} + \imath\mathfrak{v} = \Pi(-\mathfrak{q} + \imath\mathfrak{v}).$$



Reformulations of the scattering problem

Find $u_h \in \mathrm{V}_h(\Omega)$ and
$a(u_h, v_h) = \ell(v_h) \quad \forall v_h \in \mathrm{V}_h(\Omega)$

$$a(u,v) := \sum_{j=1}^{\mathrm{J}} \int_{\Omega_j} \nabla u \cdot \nabla \overline{v} - \kappa^2 u \overline{v}\, d\boldsymbol{x} - \int_{\partial\Omega_j} \imath\kappa\, u \overline{v}\, d\sigma$$

$$\ell(v) := \sum_{j=1}^{\mathrm{J}} \int_{\Omega_j} f\overline{v} d\boldsymbol{x} \qquad u, v \in \mathbb{V}_h(\Omega)$$

$$\Updownarrow$$

Find $u_h \in \mathbb{V}_h(\Omega)$, $\mathfrak{p}_h \in \mathbb{V}_h(\Sigma)$ and $\forall v_h \in \mathbb{V}_h(\Omega)$

$$a(u_h, v_h) - \imath t_h(u_h|_\Sigma, v_h|_\Sigma) = t_h(\mathfrak{p}_h, v_h|_\Sigma) + \ell(v_h)$$

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Non trivial theorem :

- relaxing constraints with Lagrange multipliers
- Use Π to enforce continuity

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$\Longleftrightarrow$

Skeleton formulation
$\mathfrak{p}_h \in \mathbb{V}_h(\Sigma)$ and
$(\mathrm{Id} + \Pi\mathrm{S})\mathfrak{p}_h = g_h$

Unknowns u_h are eliminated in all subdomains in parallel by local "ingoing -to-outgoing" solves, applying a (block diagonal) scattering operator.

Proposition : Define $\mathrm{S}(\mathfrak{p}) := \mathfrak{p} + 2\imath w|_\Sigma$ where $w \in \mathbb{V}_h(\Omega)$ satisfies $a(w, v) - \imath t_h(w|_\Sigma, v|_\Sigma) = t_h(\mathfrak{p}, v|_\Sigma) \forall v \in \mathbb{V}_h(\Omega)$. Then $\|\mathrm{S}(\mathfrak{p})\|_{t_h} \leq \|\mathfrak{p}\|_{t_h}$ for all $\mathfrak{p} \in \mathbb{V}_h(\Sigma)$.

Convergence estimate

Theorem :

1) Boundedness : $\|\mathrm{Id} + \Pi\mathrm{S}\|_{t_h} \leq 2$

2) Coercivity : $\Re e\{t_h(\mathfrak{v}, (\mathrm{Id} + \Pi\mathrm{S})\mathfrak{v})\} \geq \gamma_h^2 \|\mathfrak{v}\|_{t_h}^2 \quad \forall \mathfrak{v} \in \mathbb{V}_h(\Sigma)$

Coercivity constant

$$\gamma_h := \frac{\alpha}{\lambda_h^+ + 2\, C_{\mathrm{sz}}\, \|a\| / \lambda_h^-}$$

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$$\alpha := \inf_{u_h, v_h \in \mathrm{V}_h(\Omega)\setminus\{0\}} \sup \frac{|a(u_h, v_h)|}{\|u_h\|_{\mathrm{H}^1(\Omega)} \|v_h\|_{\mathrm{H}^1(\Omega)}}$$

Coercivity constant

continuity modulus of Scott-Zhang interpolator

$$\gamma_h := \frac{\alpha}{\lambda_h^+ + 2\, C_{\mathrm{sz}}\, \|a\| / \lambda_h^-}$$

$$\|a\| := \sup_{u,v \in \mathrm{H}^1(\Omega)\setminus\{0\}} \sup \frac{|a(u,v)|}{\|u\|_{\mathrm{H}^1(\Omega)} \|v\|_{\mathrm{H}^1(\Omega)}}$$

Convergence estimate

Theorem :

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continuity modulus of Scott-Zhang interpolator

Coercivity constant

$$\|a\| := \sup_{u,v \in \mathrm{H}^1(\Omega)\setminus\{0\}} \sup \frac{|a(u,v)|}{\|u\|_{\mathrm{H}^1(\Omega)} \|v\|_{\mathrm{H}^1(\Omega)}}$$

$$\gamma_h := \frac{\alpha}{\lambda_h^+ + 2\, C_{\mathrm{sz}}\, \|a\| / \lambda_h^-}$$

$$\lambda_h^+ := \sup_{\substack{j=1\dots \mathrm{J} \\ v_h \in \mathrm{V}_h(\Gamma_j)\setminus\{0\}}} \frac{t_{\Gamma_j}(v_h, v_h)}{\|v_h\|^2_{\mathrm{H}^{1/2}(\Gamma_j)}} \qquad \lambda_h^- := \inf_{\substack{j=1\dots \mathrm{J} \\ v_h \in \mathrm{V}_h(\Gamma_j)\setminus\{0\}}} \frac{t_{\Gamma_j}(v_h, v_h)}{\|v_h\|^2_{\mathrm{H}^{1/2}(\Gamma_j)}}$$

Convergence estimate

Theorem :

1) Boundedness : $\|\mathrm{Id} + \Pi\mathrm{S}\|_{t_h} \leq 2$

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The exact solution $\mathfrak{p}^{(\infty)} \in \mathbb{V}_h(\Sigma)$ to the skeleton formulation can be computed with e.g. a Richardson iteration : given $r \in (0,1)$, compute

$$\mathfrak{p}^{(n+1)} = (1-r)\mathfrak{p}^{(n)} - r\Pi\mathrm{S}\mathfrak{p}^{(n)} + rg_h.$$

Proposition : convergence of Richardson's solver

$$\frac{\|\mathfrak{p}^{(n)} - \mathfrak{p}^{(\infty)}\|_{t_h}}{\|\mathfrak{p}^{(0)} - \mathfrak{p}^{(\infty)}\|_{t_h}} \leq \left(\, 1 - 2r(1-r)\gamma_h^2 \,\right)^{n/2}.$$

Important consequence : If the $t_{\Gamma_j}(\cdot,\cdot)$'s yield norms that are $h-$uniformly equivalent to $\|\cdot\|_{\mathrm{H}^{1/2}(\Gamma_j)}$, then we have $h-$uniform geometric convergence.

Outline

I Review of the Optimized Schwarz Method

II New manner to enforce transmission conditions

III Numerical results

Numerical experiments : Helmholtz in 2D

Constant wave number $\kappa > 0$ in a disc $\Omega = \mathrm{D}(0,1)$ and impedance boundary condition $(\partial_{\boldsymbol{n}} - \imath\kappa)u^{\mathrm{ex}} = g$ with $g(\boldsymbol{x}) = (\partial_{\boldsymbol{n}} - \imath\kappa)\exp(-\imath\kappa\mathbf{d}\cdot\boldsymbol{x})$, discretization with $\mathrm{V}_h(\Omega) = \mathbb{P}_1$-Lagrange.

$$u_h^{\mathrm{ex}} \in \mathrm{V}_h(\Omega) \quad \text{and} \quad a(u_h^{\mathrm{ex}}, v_h) = \ell(v_h) \quad \forall v_h \in \mathrm{V}_h(\Omega)$$

$$\begin{aligned} &a(u,v) = \int_\Omega \nabla u\cdot\nabla\overline{v} - \kappa^2 u\overline{v}d\boldsymbol{x} - \imath\kappa\int_{\partial\Omega} u\overline{v}d\sigma \\ &\ell(v) = \int_{\partial\Omega}\overline{v}gd\sigma. \end{aligned}$$

With $u_0^{(0)} \equiv 0$, we denote $u_h^{(n)}$ the iterates of the linear solver. The measured error is given by

$$(\text{relative error})^2 = \frac{\sum_{j=1}^{\mathrm{J}} \|u_h^{(n)} - u_h^{\mathrm{ex}}\|^2_{\mathrm{H}^1(\Omega_j)}}{\sum_{j=1}^{\mathrm{J}} \|u_h^{(0)} - u_h^{\mathrm{ex}}\|^2_{\mathrm{H}^1(\Omega_j)}}.$$

Remarks :

- global linear solver is GMRes, relative tolerance $= 10^{-8}$
- sequential computations on a 6 core workstation
- FEM & DDM code `NIDDL` (in `Julia`) + `BemTool` (in C++) for integral operators
- exchange operator Π computed with PCG.

Choices of impedance

Recall that $t_h(\mathfrak{p}, \mathfrak{q}) = t_{\partial\Omega_1}(p_1, q_1) + \cdots + t_{\partial\Omega_J}(p_J, q_J)$. We tested several choices of local impedances.

Choice 1 : M = **surface mass matrix**
$t_{\partial\Omega_j}(p_h, q_h) = \int_{\partial\Omega_j} p_h(\boldsymbol{x})\overline{q}_h(\boldsymbol{x})d\sigma(\boldsymbol{x})$
This is the impedance originally considered by Després.

Choice 2 : K = **surface** $\mathrm{H1}$**-scalar product**
$t_{\partial\Omega_j}(p_h, q_h) = \int_{\partial\Omega_j} \kappa^{-1}\nabla p_h(\boldsymbol{x}) \cdot \nabla\overline{q}_h(\boldsymbol{x})/2 + \kappa\, p_h(\boldsymbol{x})\overline{q}_h(\boldsymbol{x})d\sigma(\boldsymbol{x})$

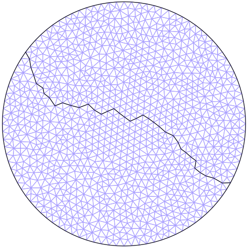
Choice 3 : W = **positive hypersingular integral operator**

$$t_{\partial\Omega_j}(p_h, q_h) = \int_{\partial\Omega_j\times\partial\Omega_j} \exp(-\kappa|\boldsymbol{x}-\boldsymbol{y}|)/(4\pi|\boldsymbol{x}-\boldsymbol{y}|)[\kappa^{-1}\, \boldsymbol{n}(\boldsymbol{x}) \times \nabla_{\partial\Omega_j} p_h(\boldsymbol{x}) \cdot \boldsymbol{n}(\boldsymbol{y}) \times \nabla_{\partial\Omega_j} q_h(\boldsymbol{y}) + \kappa\, \boldsymbol{n}(\boldsymbol{x}) \cdot \boldsymbol{n}(\boldsymbol{y}) p_h(\boldsymbol{x}) q_h(\boldsymbol{y})\,]d\sigma(\boldsymbol{x}, \boldsymbol{y})$$

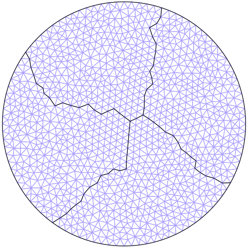
Choice 4 : Λ = **schur complement**($\simeq$ discrete DtN) associated with the interior numerical solution to the **positive** problem $-\Delta v + \kappa^2 v = 0$ in Ω_j.

Mesh partitionning

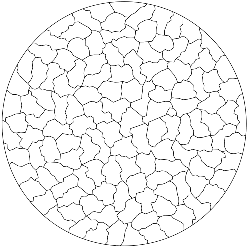
Meshes were generated a priori on the whole computational domain with GMSH. Partitionning is obtained a posteriori with `Metis`.



2 subdomains



4 subdomains

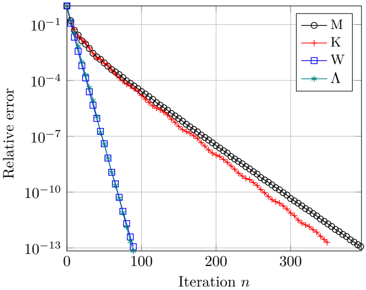


128 subdomains

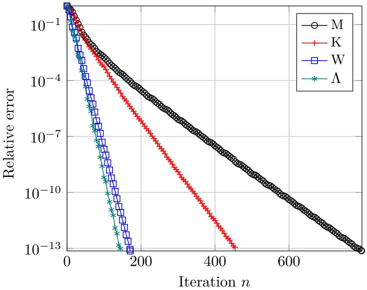
Convergence history

$\kappa = 5, \quad \lambda = 2\pi/\kappa \simeq 1.25$
$N_\lambda = \lambda/h = 40$ points/wavelength.

4 subdomains

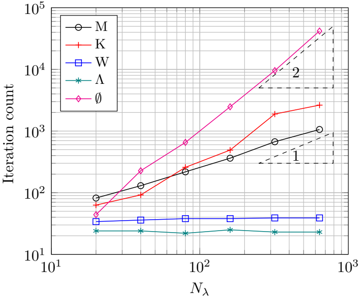


128 subdomains



Iteration count vs. N_λ = points/wavelength

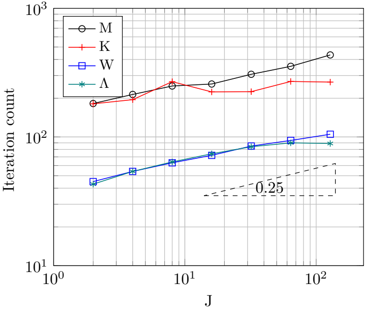
$\kappa = 1$, $\lambda = 2\pi/\kappa \simeq 6.28$, $N_\lambda = \lambda/h$, 4 subdomains.
Relative tolerance of GMRes $= 10^{-8}$, $\emptyset$ = no DDM.



N_λ	$\emptyset$	M	K	W	Λ
20	44	82	63	34	24
40	227	130	92	36	24
80	654	218	258	38	22
160	2474	363	491	38	25
320	9559	671	1888	39	23
640	41888	1060	2633	39	23

Iteration count vs. J = number of subdomains

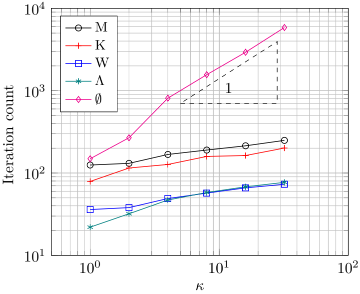
$\kappa = 5, \quad \lambda = 2\pi/\kappa \simeq 1.26, \quad N_\lambda = \lambda/h = 40.$
Relative tolerance of GMRes $= 10^{-8}$.



J	M	K	W	Λ
2	182	181	45	43
4	214	195	54	54
8	249	269	63	64
16	258	224	72	74
32	307	225	85	84
64	354	270	94	90
128	434	267	105	89

Iteration count vs. $\kappa =$ wavenumber

$\lambda = 2\pi/\kappa$, $N_\lambda = \lambda/h = 30$, 4 subdomains.
Relative tolerance of GMRes $= 10^{-8}$, $\emptyset$ =no DDM.



κ	$\emptyset$	M	K	W	Λ
1	149	125	79	36	22
2	268	131	115	38	32
4	811	168	127	49	47
8	1563	190	159	57	58
16	2926	214	163	66	68
32	5846	249	201	73	77

Conclusion

We proposed a new way of imposing transmission conditions involving another choice of exchange operator. This yields *h*-uniform convergence of iterative solvers, and accomodates cross-points.

In addition this approach appears as a natural generalization of the classical OSM à la Després, and allows to propose a full theoretical framework, which was not available so far.

Also available :

- other boundary conditions (Dirichlet, Neumann),
- other equations (3D Helmholtz, Maxwell),
- analysis of non-infsup-stable impedances.

Future investigations

- fine properties of the exchange operator
- large scale optimized parallel implementation
- multi-level strategy
- non-conforming DDM

Thank you for your attention

Questions ?

X.Claeys, F.Collino, P.Joly and E.Parolin, "A discrete domain decomposition method for acoustics with uniform exponential rate of convergence using non-local impedance operators", proceedings of the DD25 conference.

X.Claeys, "A new variant of the Optimised Schwarz Method for arbitrary non-overlapping subdomain partitions”, accepted in ESAIM M2AN, preprint Arxiv 1910.05055

X.Claeys and E.Parolin, "Robust treatment of cross points in Optimized Schwarz Methods", submitted, preprint Arxiv 2003.06657