Generalized Optimized Schwarz Methods in arbitrary non-overlapping subdomain partitions

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joint work with
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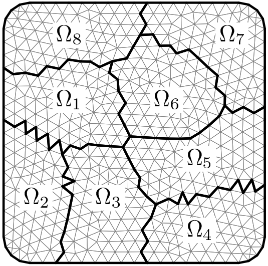
Classical acoustic scattering problem

frequency : $\omega > 0$
source : $f \in \mathrm{L}^2(\Omega)$

Non-overlapping partition
$\Omega = \overline{\Omega}_1 \cup \cdots \cup \overline{\Omega}_{\mathrm{J}}$,
$\Gamma_j := \partial\Omega_j, \quad \Gamma'_j := \Gamma_j \setminus \partial\Omega$
Ω_j : Lipschitz, bounded



Helmholtz bvp
$\Delta u + \omega^2 u = -f$ in Ω,
$\partial_{\boldsymbol{n}} u - \imath\omega\, u = 0$ on $\partial\Omega$.

$\Longleftrightarrow$

local sub-problems $j = 1 \dots \mathrm{J}$
$\Delta u + \omega^2 u = -f$ in Ω_j
$\partial_{\boldsymbol{n}} u - \imath\omega\, u = 0$ on $\partial\Omega_j \cap \partial\Omega$.

+

transmission conditions
$\partial_{n_j} u|_{\Gamma_j}^{\mathrm{int}} = -\partial_{n_k} u|_{\Gamma_k}^{\mathrm{int}}$
$u|_{\Gamma_j}^{\mathrm{int}} = u|_{\Gamma_k}^{\mathrm{int}}$ on $\Gamma_j \cap \Gamma_k, \forall j, k$

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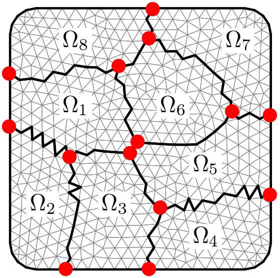

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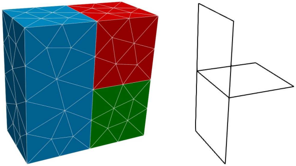
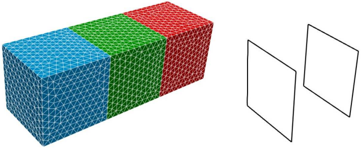
What are cross-points ?

These are points where either three subdomains are adjacent **or** two subdomains meet at the external boundary. They form the boundaries of interfaces :

$$\{\text{cross-points}\} = \bigcup_{j \neq k} \partial(\Gamma_j \cap \Gamma_k) = \text{"wire basket"}$$

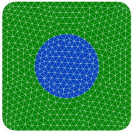


Example in 3D



Example in 3D (waveguide config)

Example in 2D
no cross-point



Outline

I Review of the Optimized Schwarz Method

II New manner to enforce transmission conditions

III Numerical results

Original Després algorithm

Transmission conditions :

$$\begin{aligned} \partial_{n_j} u|_{\Gamma_j} &= -\partial_{n_k} u|_{\Gamma_k} \\ u|_{\Gamma_j} &= u|_{\Gamma_k} \end{aligned} \iff \begin{aligned} -\partial_{n_j} u|_{\Gamma_j} + \imath\omega u|_{\Gamma_j} &= \\ +\partial_{n_k} u|_{\Gamma_k} + \imath\omega u|_{\Gamma_k} & \end{aligned} \iff \begin{aligned} (-\partial_{n_j} u|_{\Gamma_j'} + \imath\omega u|_{\Gamma_j'})_{j=1}^{\mathrm{J}} &= \\ \Pi_0((+\partial_{n_k} u|_{\Gamma_k'} + \imath\omega u|_{\Gamma_k'})_{k=1}^{\mathrm{J}}) & \end{aligned}$$

on $\Gamma_j \cap \Gamma_k \; \forall j, k$ on $\Gamma_j \cap \Gamma_k \; \forall j, k$

where Π_0 swaps traces on interfaces :
$(v_0, \dots, v_{\mathrm{J}}) = \Pi_0(u_0, \dots, u_{\mathrm{J}}) \iff v_j = u_k$ on $\Gamma_j \cap \Gamma_k$.

Local scattering operators :

$\mathrm{S}_0^{\Gamma_j}(-\partial_{n_j}\psi|_{\Gamma_j'} + \imath\omega\psi|_{\Gamma_j'}) := \partial_{n_j}\psi|_{\Gamma_j'} + \imath\omega\psi|_{\Gamma_j'}$
for $\Delta\psi + \omega^2\psi = 0$ in Ω_j
$\partial_{\boldsymbol{n}}\psi - \imath\omega\psi = 0$ on $\partial\Omega_j \cap \partial\Omega$

$$(+\partial_{n_j} u|_{\Gamma_j'} + \imath\omega u|_{\Gamma_j'})_{j=1}^{\mathrm{J}} = \mathrm{S}_0((-\partial_{n_k} u|_{\Gamma_k'} + \imath\omega u|_{\Gamma_k'})_{k=1}^{\mathrm{J}}) + g$$

with $\mathrm{S}_0 := \mathrm{diag}_{j=1\dots\mathrm{J}}(\mathrm{S}_0^{\Gamma_j})$.

B.Després, "Méthodes de décomposition de domaine pour les problèmes de propagation d'ondes en régime harmonique. Le théorème de Borg pour l'équation de Hill vectorielle." Thèse, Univ. Paris IX (Dauphine), Paris, 1991.

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stems from the source term of the bvp

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Optimized Schwarz

$$(\mathrm{Id} - \Pi_0\mathrm{S}_0)p = \Pi_0(g)$$

with $p = (-\partial_{n_j} u|_{\Gamma_j'} + \imath\omega u|_{\Gamma_j'})_{j=1}^{\mathrm{J}}$

"Parallelization friendly" equation :

i) S_0 is block-diagonal
ii) Π_0 is sparse and explicit

Contraction properties

For $p \in \mathbb{L}^2(\Sigma) := \mathrm{L}^2(\Gamma_1) \times \cdots \times \mathrm{L}^2(\Gamma_\mathrm{J})$,

i) $\|\Pi_0(p)\|_{\mathbb{L}^2(\Sigma)} = \|p\|_{\mathbb{L}^2(\Sigma)}$

ii) $\|\mathrm{S}_0(p)\|_{\mathbb{L}^2(\Sigma)} \leq \|p\|_{\mathbb{L}^2(\Sigma)}$

iii) $\Re e\{(p, (\mathrm{Id} - \Pi_0\mathrm{S}_0)p)_{\mathbb{L}^2(\Sigma)}\} \geq 0$

$\mathrm{Id} - \Pi_0\mathrm{S}_0 : \mathbb{L}^2(\Sigma) \to \mathbb{L}^2(\Sigma)$ is

- injective
- not surjective, not coercive.

Contractivity is at the core of existing convergence proofs of linear solvers such as Richardson's algorithm (with $0 < r < 1$) :

$$p^{(n+1)} = (1-r)p^{(n)} + r(\Pi_0\mathrm{S}_0)p^{(n)} + r\Pi_0(g).$$

In practice, convergence speed is spoiled by the lack of coercivity and is at best algebraic ($\gamma > 0$) :

$$\|p^{(n)} - p\|_{\mathbb{L}^2(\Sigma)} = \mathcal{O}(n^{-\gamma}).$$

Boosting convergence by tuning impedance

Basic idea : rewriting transmission conditions with a smart choice of impedance operator

$$\begin{matrix} \partial_{n_j} u|_{\Gamma_j} = -\partial_{n_k} u|_{\Gamma_k} \\ u|_{\Gamma_j} = u|_{\Gamma_k} \\ \text{on } \Gamma_j \cap \Gamma_k \ \forall j,k \end{matrix} \iff \begin{matrix} -\partial_{n_j} u|_{\Gamma_j} + \imath \mathrm{T}_{jk}(u|_{\Gamma_j}) = \\ +\partial_{n_k} u|_{\Gamma_k} + \imath \mathrm{T}_{jk}(u|_{\Gamma_k}) \\ \text{on } \Gamma_j \cap \Gamma_k \ \forall j,k \end{matrix}$$

A proper choice for T_{jk} significantly improves convergence of linear solvers. Many choices investigated :

$\mathrm{T}_{jk} =$ 2nd order surface diff. op.	[Gander, Magoules & Nataf, 2002]
$\mathrm{T}_{jk} =$ Pade approx. of DtN maps	[Boubendir, Antoine & Geuzaine, 2012]
$\mathrm{T}_{jk} =$ integral operators	[Collino, Ghanemi & Joly, 2000]

[Collino, Ghanemi & Joly] geometric convergence (contractivity + coercivity) under two conditions

(1) **no cross-point**

(2) T_{jk} induces a scalar product on $\mathrm{H}^{+1/2}(\Gamma_j \cap \Gamma_k)$.

Cross-point issue

Unappropriate treatment of cross-points may prevent convergence [Gander & Kwok, 2013]. The root cause seems related to Π_0 not being continuous in proper trace norms.

- in 2D : [Gander & Santugini, 2016], [Després & al, 2020]
- checkerboard config : [Modave & al, 2019 & 2020]
- FETI dual-primal : [Farhat & al, 2005], [Bendali & Boubendir, 2006]

No (quantitative) convergence theory available, even for specific geometrical configurations.

We propose a generic approach applicable with any subdomain partition as well as a complete theoretical framework with convergence estimates.

Idea : guided by Multi-Trace Formalism replace Π_0 by non-local counterpart Π that remains continuous no matter the presence of cross points.

X.Claeys, "Quasi-local multi-trace boundary integral formulations", Numer. Methods Partial Differential Equations, 31(6) :2043–2062, 2015.

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Discrete function spaces

Triangulation conforming with Ω_j's, FE spaces $\mathrm{V}_h(\Omega_j) = \{\ \mathbb{P}_k$-Lagrange on $\Omega_j\}$.

Volume functions	**Tuples of traces ($\boldsymbol{\Gamma_j := \partial\Omega_j}$)**
$\mathbb{H}_h(\Omega) := \mathrm{V}_h(\Omega_1) \times \cdots \times \mathrm{V}_h(\Omega_\mathrm{J})$	$\mathbb{H}_h(\Sigma) := \mathrm{V}_h(\Gamma_1) \times \cdots \times \mathrm{V}_h(\Gamma_\mathrm{J})$
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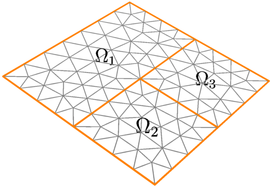

Discrete function spaces

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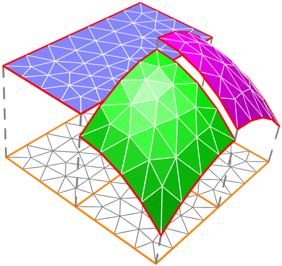
Piecewise H^1
Possible jumps through interfaces



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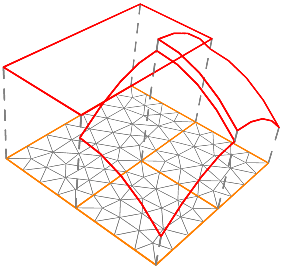
Dirichlet multi-traces = $\mathbb{H}_h(\Sigma)$
tuples of traces at boundaries

Neumann multi-traces = $\mathbb{H}_h(\Sigma)^*$



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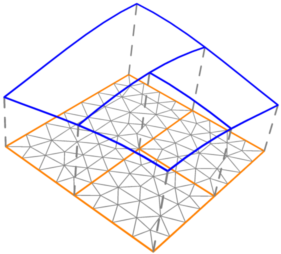
Dirichlet single-traces = $\mathrm{V}_h(\Sigma)$
tuples that match at interfaces

Neumann single-traces = $\mathrm{V}_h(\Sigma)^\circ$
$\{\mathfrak{p} \in \mathbb{H}_h(\Sigma)^*, \langle \mathfrak{p}, \mathfrak{v} \rangle = 0\ \forall \mathfrak{v} \in \mathrm{V}_h(\Sigma)\}$
= polar set of $\mathrm{V}_h(\Sigma)$.



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Discrete trace operator : $\mathrm{B}(u_1,\dots,u_{\mathrm{J}}) := (u_1|_{\Gamma_1},\dots,u_{\mathrm{J}}|_{\Gamma_{\mathrm{J}}})$. Globally continous FE functions are caracterized by continuity constraints at interfaces

$$\mathrm{V}_h(\Omega) = \{\mathfrak{u} \in \mathbb{H}_h(\Omega),\ \mathrm{B}(\mathfrak{u}) \in \mathrm{V}_h(\Sigma)\}$$

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$$\begin{aligned} \mathrm{V}_h(\Omega) &= \{\mathfrak{u} \in \mathbb{H}_h(\Omega),\ \mathrm{B}(\mathfrak{u}) \in \mathrm{V}_h(\Sigma)\} \\ &= \{\mathfrak{u} \in \mathbb{H}_h(\Omega),\ \langle \mathrm{B}(\mathfrak{u}), \mathfrak{q}\rangle = 0\ \forall \mathfrak{q} \in \mathrm{V}_h(\Sigma)^\circ\} \end{aligned}$$

Discrete function spaces

Triangulation conforming with Ω_j's, FE spaces $\mathrm{V}_h(\Omega_j) = \{\, \mathbb{P}_k$-Lagrange on $\Omega_j\}$.

Volume functions	**Tuples of traces ($\boldsymbol{\Gamma_j := \partial\Omega_j}$)**
$\mathbb{H}_h(\Omega) := \mathrm{V}_h(\Omega_1)\times\cdots\times\mathrm{V}_h(\Omega_{\mathrm{J}})$	$\mathbb{H}_h(\Sigma) := \mathrm{V}_h(\Gamma_1)\times\cdots\times\mathrm{V}_h(\Gamma_{\mathrm{J}})$
$\mathrm{V}_h(\Omega) := \{\,(u_1,\dots,u_{\mathrm{J}})\in\mathbb{H}_h(\Omega),\; u_j = u_k \text{ on } \Gamma_j\cap\Gamma_k\;\forall j,k\,\}$	$\mathrm{V}_h(\Sigma) := \{\,(p_1,\dots,p_{\mathrm{J}})\in\mathbb{H}_h(\Sigma),\; p_j = p_k \text{ on } \Gamma_j\cap\Gamma_k\;\forall j,k\,\}$

Discrete trace operator : $\mathrm{B}(u_1,\dots,u_{\mathrm{J}}) := (u_1|_{\Gamma_1},\dots,u_{\mathrm{J}}|_{\Gamma_{\mathrm{J}}})$. Globally continous FE functions are caracterized by continuity constraints at interfaces

$$\begin{aligned}\mathrm{V}_h(\Omega) &= \{\mathfrak{u}\in\mathbb{H}_h(\Omega),\ \mathrm{B}(\mathfrak{u})\in\mathrm{V}_h(\Sigma)\}\\ &= \{\mathfrak{u}\in\mathbb{H}_h(\Omega),\ \langle\mathfrak{u},\mathrm{B}^*(\mathfrak{q})\rangle = 0\ \forall\mathfrak{q}\in\mathrm{V}_h(\Sigma)^\circ\}\end{aligned}$$

Generalized exchange operator

Impedance matrix $\mathrm{T} : \mathbb{H}_h(\Sigma) \to \mathbb{H}_h(\Sigma)^*$

$$\Re e\{\langle \mathrm{T}(\mathfrak{p}), \overline{\mathfrak{p}}\rangle\} > 0 \;\; \forall \mathfrak{p} \neq 0$$

- Typically $\mathrm{T} = \mathrm{diag}(\mathrm{T}_1, \ldots, \mathrm{T}_\mathrm{J})$
- Convergence measured with the scalar product $\mathrm{T}_\mathrm{s}^{-1} := 2(\mathrm{T} + \mathrm{T}^*)^{-1}$

Examples of impedance :

- surface mass matrix,
- surface order 2 operator,
- layer potential, DtN map,
- Schur complements, etc...

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Theorem : for $\mathfrak{p} \in \mathbb{H}_h(\Sigma)^*$ define

$\Pi(\mathfrak{p}) := (\mathrm{T} + \mathrm{T}^*)\mathfrak{u} - \mathfrak{p}$ where
$\mathfrak{u} \in \mathrm{V}_h(\Sigma)$, $\langle \mathrm{T}^*\mathfrak{u}, \mathfrak{v}\rangle = \langle \mathfrak{p}, \mathfrak{v}\rangle\ \forall \mathfrak{v} \in \mathrm{V}_h(\Sigma)$.

This is an isometry $\|\Pi(\mathfrak{p})\|_{\mathrm{T}_s^{-1}} = \|\mathfrak{p}\|_{\mathrm{T}_s^{-1}}$.

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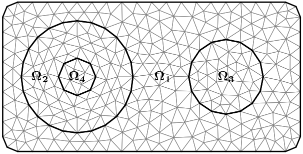
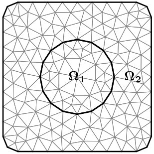
Remark : If $\mathrm{T} = \mathrm{T}^*$ then Π is a T^{-1}-orthogonal symetry and $\Pi^2 = \mathrm{Id}$.

Theorem : Recall that $\mathrm{V}_h(\Sigma)^\circ :=$ polar set of $\mathrm{V}_h(\Sigma)$.
For $(\mathfrak{u}, \mathfrak{p}) \in \mathbb{H}_h(\Sigma) \times \mathbb{H}_h(\Sigma)^*$,

$$(\mathfrak{u}, \mathfrak{p}) \in \mathrm{V}_h(\Sigma) \times \mathrm{V}_h(\Sigma)^\circ \quad \Longleftrightarrow \quad -\mathfrak{p} + \imath \mathrm{T}(\mathfrak{u}) = \Pi(\,\mathfrak{p} + \imath \mathrm{T}^*(\mathfrak{u})\,)$$

In case of no cross point





Lemma : Assume

1) geometric configuration involves no cross point,
2) impedance does not couple disjoint connected components of the skeleton,
3) impedance operators are adjoint to each other on both sides of interfaces.

Then the exchange operator reduces to local swaps :

$$\Pi = \Pi_0 = \text{local swaps}$$

- In this situation, our DDM strategy coincides with Després' algorithm.
- Such conditions are satisfied by all usual impedance operators.
  Examples : Després, 2nd order tangential operator, etc...

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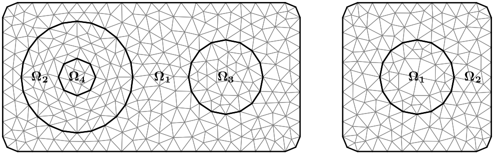

Equivalently rewritten :
$\Pi_0 \mathrm{T} = \mathrm{T}^* \Pi_0^*$

[Collino, Joly & Lecouvez,2020]

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When is exchange operator reduced to swaps ?

The operation $\mathfrak{p} \mapsto \Pi(\mathfrak{p})$ is non-trivial and potentially costly, except if $\Pi = \Pi_0$. For the general case with cross-points, when does this happen ? For which choice of impedance ?

Lemma 1 : $\Pi_0 \mathrm{T} = \mathrm{T}^* \Pi_0^* \implies \Pi = \Pi_0$.

Lemma 2 :
Define $\Theta(\mathrm{T}) := (\mathrm{T} + \Pi_0 \mathrm{T} \Pi_0^*)/2$. Then $\Theta \circ \Theta = \Theta$ and the condition of **Lemma 1** is satisfied if and only if $\mathrm{T} = \Theta(\mathrm{T})$.

Similar condition in :

Despres, Nicolopoulos & Thierry, "On Domain Decomposition Methods with optimized transmission conditions and cross-points", preprint hal-03230250 (2021).

Skeleton formulation

Find $u_h \in \mathrm{V}_h(\Omega)$ and

$a(u_h, v_h) = \ell(v_h) \quad \forall v_h \in \mathrm{V}_h(\Omega)$

$$\ell(v_h) := \int_\Omega f \overline{v}_h d\boldsymbol{x}$$

$$a(u_h, v_h) := \int_\Omega \nabla u_h \cdot \nabla v_h - \omega^2 u_h \overline{v}_h \, d\boldsymbol{x}$$

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$$= \langle \mathrm{A} \cdot (u_h|_{\Omega_1}, \dots, u_h|_{\Omega_\mathrm{J}}), (v_h|_{\Omega_1}, \dots, v_h|_{\Omega_\mathrm{J}}) \rangle$$

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$$-\mathfrak{p} + \imath\mathrm{T}\,\mathrm{B}\mathfrak{u} = \Pi(\mathfrak{p} + \imath\mathrm{T}^*\mathrm{B}\mathfrak{u})$$

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$-\mathfrak{p} + \imath \mathrm{T}\,\mathrm{B}\mathfrak{u} = \Pi(\mathfrak{p} + \imath \mathrm{T}^* \mathrm{B}\mathfrak{u})$

$$\Longleftrightarrow$$

Find $\mathfrak{q} \in \mathbb{H}_h(\Sigma)^*$
$(\mathrm{Id} - \Pi \mathrm{S})\mathfrak{q} = \mathfrak{f}$

with $\mathfrak{q} = -\mathfrak{p} + \imath \mathrm{T} \mathrm{B} \mathfrak{u}$

Volume unkowns $\mathfrak{u}$ are eliminated in all subdomains in parallel by local "ingoing-to-outgoing" solves, applying a (block diagonal) scattering operator.

Proposition : The scattering operator $\mathrm{S} := -\mathrm{Id} - \imath(\mathrm{T} + \mathrm{T}^*)\mathrm{B}(\mathrm{A} - \imath \mathrm{B}^* \mathrm{T} \mathrm{B})^{-1}\mathrm{B}^*$ is a contraction $\|\mathrm{S}(\mathfrak{p})\|_{\mathrm{T}_s^{-1}} \leq \|\mathfrak{p}\|_{\mathrm{T}_s^{-1}}\ \forall \mathfrak{p} \in \mathbb{H}_h(\Sigma)^*$.

Convergence estimate

Theorem :

1) Boundedness : $\|\mathrm{Id} - \Pi\mathrm{S}\|_{\mathrm{T_s}^{-1}} \leq 2$

2) Coercivity : $\Re e\{\langle \mathrm{T_s}^{-1}(\mathrm{Id} - \Pi\mathrm{S})\mathfrak{v}, \overline{\mathfrak{v}}\rangle\} \geq \gamma_h^2 \|\mathfrak{v}\|^2_{\mathrm{T_s}^{-1}} \quad \forall \mathfrak{v} \in \mathbb{H}_h(\Sigma)^*$

Coercivity constant

$$\gamma_h := \frac{\alpha}{\lambda_h^+ + 2\,\|a\|/\lambda_h^-}\,\frac{2}{1 + (1+\mu)^2}$$

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$$\mu = \sup_{\mathfrak{v}\in\mathbb{H}_h(\Sigma)\setminus\{0\}} \left|\frac{\langle(\mathrm{T}-\mathrm{T}^*)\mathfrak{v},\overline{\mathfrak{v}}\rangle}{\langle(\mathrm{T}+\mathrm{T}^*)\mathfrak{v},\overline{\mathfrak{v}}\rangle}\right|$$

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$$\alpha := \inf_{u_h, v_h \in \mathrm{V}_h(\Omega)\setminus\{0\}} \sup \frac{|a(u_h, v_h)|}{\|u_h\|_{\mathrm{H}^1_\omega(\Omega)} \|v_h\|_{\mathrm{H}^1_\omega(\Omega)}}$$

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$$\lambda_h^+ := \sup_{\mathfrak{v} \in \mathbb{H}_h(\Sigma)\setminus\{0\}} \frac{\langle \mathrm{T_s}(\mathfrak{v}), \overline{\mathfrak{v}}\rangle}{\langle \Lambda(\mathfrak{v}), \overline{\mathfrak{v}}\rangle}$$

$$\mu = \sup_{\mathfrak{v} \in \mathbb{H}_h(\Sigma)\setminus\{0\}} \left| \frac{\langle (\mathrm{T} - \mathrm{T}^*)\mathfrak{v}, \overline{\mathfrak{v}}\rangle}{\langle (\mathrm{T} + \mathrm{T}^*)\mathfrak{v}, \overline{\mathfrak{v}}\rangle} \right|$$

$\Lambda =$ reference impedance

$$\lambda_h^- := \inf_{\mathfrak{v} \in \mathbb{H}_h(\Sigma)\setminus\{0\}} \frac{\langle \mathrm{T_s}(\mathfrak{v}), \overline{\mathfrak{v}}\rangle}{\langle \Lambda(\mathfrak{v}), \overline{\mathfrak{v}}\rangle}$$

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The exact solution $\mathfrak{p}^{(\infty)} \in \mathbb{H}_h(\Sigma)^*$ to the skeleton formulation can be computed with e.g. a Richardson iteration (with $0 < r < 1$)

$$\mathfrak{p}^{(n+1)} = (1 - r)\mathfrak{p}^{(n)} + r(\Pi\mathrm{S})\mathfrak{p}^{(n)} + r\mathfrak{f}.$$

Proposition : convergence of Richardson's solver

$$\frac{\|\mathfrak{p}^{(n)} - \mathfrak{p}^{(\infty)}\|_{\mathrm{T}_s^{-1}}}{\|\mathfrak{p}^{(0)} - \mathfrak{p}^{(\infty)}\|_{\mathrm{T}_s^{-1}}} \leq \left(1 - 2r(1 - r)\gamma_h^2 \right)^{n/2}.$$

Important consequence : If $\|\ \ \|_{\mathrm{T}_s}$ is $h-$uniformly equivalent to $\|\cdot\|_{\Lambda}$, then we have $h-$uniform geometric convergence.

Reference impedance

$$\Lambda = \mathrm{diag}(\Lambda_1, \dots, \Lambda_{\mathrm{J}})$$

$$\langle \Lambda_j(v), \overline{v} \rangle := \inf\{\|\nabla \Phi\|^2_{\mathrm{H}^1_\omega(\Omega_j)} : \ \Phi \in \mathrm{V}_h(\Omega_j), \ \Phi|_{\Gamma_j} = v\}$$

Schur complements associated to H^1_ω-norms in subdomains.

This is a viable choice of impedance that leads to $\lambda_h^{\pm} = 1, \mu = 0$ and to the refined estimate :

$$\gamma_h = \sqrt{\text{coercivity cst}} \geq \frac{\alpha}{1 + 2\|a\|}.$$

If the subdomain Ω_j is fixed and the triangulation is h-uniformly shape regular, the norm induced by Λ_j is h-uniformly equivalent to the Slobodeckii norm

$$\langle \Lambda_j(v), \overline{v} \rangle \sim \int_{\Gamma_j \times \Gamma_j} \frac{|v(x) - v(y)|^2}{|x - y|^d} d\sigma(x, y).$$

The general theory only rests on 3 assumptions

H1) $\Im m\{a(v,v)\} \leq 0 \;\forall v \in \mathrm{H}^1(\Omega)$ i.e. the medium can only absorb or propagate. The case of "purely" propagative media $\Im m\{a(v,v)\} = 0$ $\forall v \in \mathrm{H}^1(\Omega)$ is covered in particular.

H2) Unique solvability of the discrete pb :

$$\inf_{u_h \in \mathrm{V}_h(\Omega)} \; \sup_{v_h \in \mathrm{V}_h(\Omega)} \frac{|a(u_h, v_h)|}{\|u_h\|_{\mathrm{H}^1_\omega(\Omega)} \|v_h\|_{\mathrm{H}^1_\omega(\Omega)}} > 0.$$

H3) Coercivity of impedance T.

No assumption on :

- the shape constant of the mesh
- the frequency regime
- impedance does not need to be non-local

Outline

I Review of the Optimized Schwarz Method

II New manner to enforce transmission conditions

III Numerical results

Numerical experiments : Helmholtz in 2D

Constant wave number $\omega > 0$ in a disc $\Omega = \mathrm{D}(0,1)$ and impedance boundary condition $(\partial_{\boldsymbol{n}} - \imath\omega)u^{\mathrm{ex}} = g$ with $g(\boldsymbol{x}) = (\partial_{\boldsymbol{n}} - \imath\omega)\exp(-\imath\omega\mathbf{d}\cdot\boldsymbol{x})$, discretization with $\mathrm{V}_h(\Omega) = \mathbb{P}_1$-Lagrange.

$$\begin{aligned}
&u_h^{\mathrm{ex}} \in \mathrm{V}_h(\Omega) \quad \text{and} \quad a(u_h^{\mathrm{ex}}, v_h) = \ell(v_h) \quad \forall v_h \in \mathrm{V}_h(\Omega)\\
&a(u,v) = \int_\Omega \nabla u\cdot\nabla\overline{v} - \omega^2 u\overline{v}d\boldsymbol{x} - \imath\omega\int_{\partial\Omega} u\overline{v}d\sigma\\
&\ell(v) = \int_{\partial\Omega}\overline{v}gd\sigma.
\end{aligned}$$

With $u_0^{(0)} \equiv 0$, we denote $u_h^{(n)}$ the iterates of the linear solver. The measured error is given by

$$(\text{relative error})^2 = \frac{\sum_{j=1}^{\mathrm{J}} \|u_h^{(n)} - u_h^{\mathrm{ex}}\|^2_{\mathrm{H}^1_\omega(\Omega_j)}}{\sum_{j=1}^{\mathrm{J}} \|u_h^{(0)} - u_h^{\mathrm{ex}}\|^2_{\mathrm{H}^1_\omega(\Omega_j)}}.$$

Remarks :

- global linear solver is GMRes, relative tolerance $= 10^{-8}$
- sequential computations on a 6 core workstation
- FEM & DDM code `NIDDL` (in `Julia`) + `BemTool` (in C++) for integral operators
- exchange operator Π computed with PCG.

Choices of impedance

We choose systematically $\langle \mathrm{T}(\mathfrak{p}), \mathfrak{q}\rangle = \langle \mathrm{T}_1(p_1), q_1\rangle + \dots + \langle \mathrm{T}_\mathrm{J}(p_\mathrm{J}), q_\mathrm{J}\rangle$.
Several choices of local impedances.

Choice 1 : $\mathrm{M} =$ **surface mass matrix** (originally considered by Després)

$$\langle \mathrm{T}_j(p_j), q_j\rangle = \int_{\Gamma_j} p_j(\boldsymbol{x})\overline{q}_j(\boldsymbol{x})d\sigma(\boldsymbol{x})$$

Choice 2 : $\mathrm{K} =$ **surface** $\mathrm{H1}$**-scalar product**

$$\langle \mathrm{T}_j(p_j), q_j\rangle = \int_{\Gamma_j} \omega^{-1}\nabla p_j(\boldsymbol{x})\cdot\nabla\overline{q}_j(\boldsymbol{x})/2 + \omega\, p_j(\boldsymbol{x})\overline{q}_j(\boldsymbol{x})d\sigma(\boldsymbol{x})$$

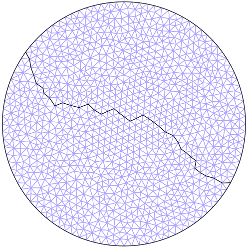
Choice 3 : $\mathrm{W} =$ **positive hypersingular integral operator**

$$\begin{aligned}\langle \mathrm{T}_j(p_j), q_j\rangle = \int_{\Gamma_j\times\Gamma_j} \exp(-\omega|\boldsymbol{x}-\boldsymbol{y}|)/(4\pi|\boldsymbol{x}-\boldsymbol{y}|)[\\ \omega^{-1}\,\boldsymbol{n}(\boldsymbol{x})\times\nabla_{\Gamma_j}p_j(\boldsymbol{x})\cdot\boldsymbol{n}(\boldsymbol{y})\times\nabla_{\Gamma_j}q_j(\boldsymbol{y})\\ +\omega\,\boldsymbol{n}(\boldsymbol{x})\cdot\boldsymbol{n}(\boldsymbol{y})p_j(\boldsymbol{x})q_j(\boldsymbol{y})\,]d\sigma(\boldsymbol{x},\boldsymbol{y})\end{aligned}$$

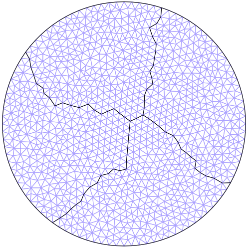
Choice 4 : $\Lambda =$ **reference impedance** ($\simeq$ discrete DtN)

Mesh partitionning

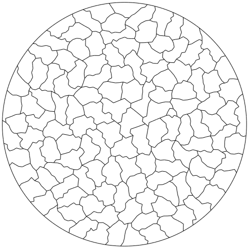
Meshes were generated a priori on the whole computational domain with `GMSH`. Partitionning is obtained a posteriori with `Metis`.



2 subdomains



4 subdomains

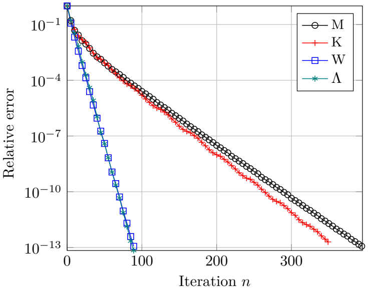


128 subdomains

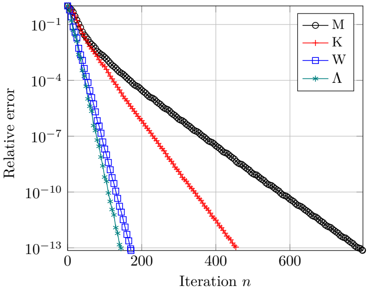
Convergence history

$\omega = 5, \quad \lambda = 2\pi/\omega \simeq 1.25$
$N_\lambda = \lambda/h = 40$ points/wavelength.

4 subdomains

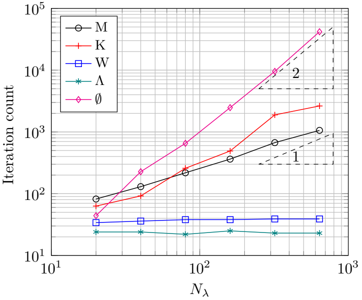


128 subdomains



Iteration count vs. N_λ = points/wavelength

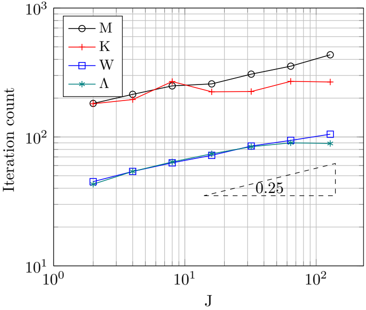
$\omega = 1$, $\lambda = 2\pi/\omega \simeq 6.28$, $N_\lambda = \lambda/h$, 4 subdomains.
Relative tolerance of GMRes $= 10^{-8}$, $\emptyset$ = no DDM.



N_λ	$\emptyset$	M	K	W	Λ
20	44	82	63	34	24
40	227	130	92	36	24
80	654	218	258	38	22
160	2474	363	491	38	25
320	9559	671	1888	39	23
640	41888	1060	2633	39	23

Iteration count vs. J = number of subdomains

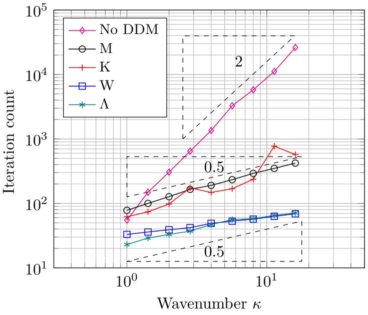
$\omega = 5, \quad \lambda = 2\pi/\omega \simeq 1.26, \quad N_\lambda = \lambda/h = 40.$
Relative tolerance of GMRes $= 10^{-8}$.



J	M	K	W	Λ
2	182	181	45	43
4	214	195	54	54
8	249	269	63	64
16	258	224	72	74
32	307	225	85	84
64	354	270	94	90
128	434	267	105	89

Iteration count vs. ω = frequency

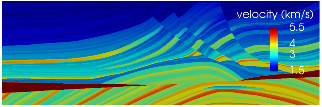
$h^2\omega^3 = (2\pi/20)^2$, 4 subdomains.
Relative tolerance of GMRes $= 10^{-8}$, $\emptyset$ =no DDM.



ω	$\emptyset$	M	K	W	Λ
1	55	78	62	33	23
2	306	127	98	39	33
4	1354	190	148	49	47
8	5791	292	237	57	58
16	26287	421	573	69	71

Helmholtz 2D Marmousi model

FreeFem + BemTool + HTool

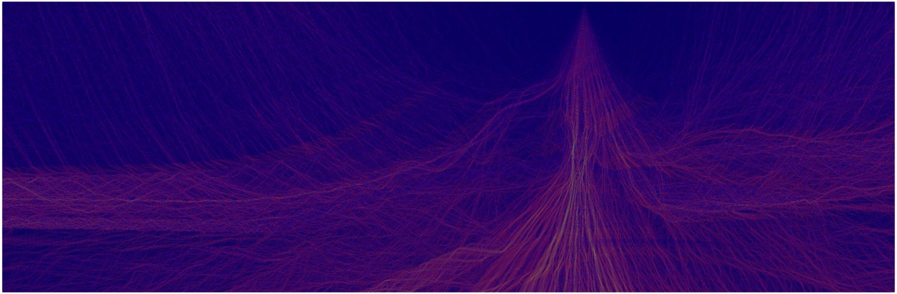


- Computation domain : 9.2×3 km
- Impedance = W = hypersingular op.
- $\mathrm{V}_h(\Omega) = \mathbb{P}_2$-Lagrange
- $N_\lambda = \lambda/h = 8$ points per wavelength
- mesh adapted to the local wavelength
- unstructured partitioning (metis)

		J			
		14	56	224	896
ω	#dofs	Iteration count			
314	3M	75	134	246	482
628	13M		137	242	464
1256	53M			264	480
4512	213M				510

Helmholtz 2D Marmousi model

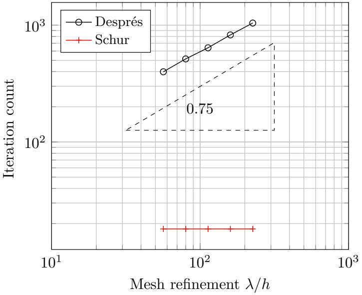
FreeFem + BemTool + HTool

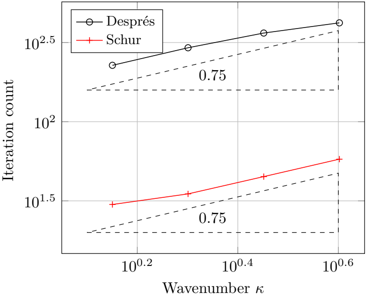


Maxwell harmonic (3D)

- Ω = unit ball
- partition (metis) : $J = 32$ (left) and $J = 16$ (right)
- discretization : Nédélec with $\omega^3 h^2 = (2\pi)^2/400$
- solved with GMRes

$$\mathbf{curl}^2 \boldsymbol{E} - \omega^2 \boldsymbol{E} = 0 \quad \text{in} \;\; \Omega$$
$$i\omega \boldsymbol{n} \times \boldsymbol{E} \times \boldsymbol{n} - \mathbf{curl}(\boldsymbol{E}) \times \boldsymbol{n} = \text{plane wave on } \partial\Omega$$





Conclusion

We proposed a new way of imposing transmission conditions involving another choice of exchange operator. For proper choices of impedance, this yields h-uniform convergence of iterative solvers, and accomodates cross-points.

In addition this approach appears as a natural generalization of the original Després algorithm, and allows to propose a full theoretical framework, which was not available so far.

Also available :

- continuous theory ($\to$ strongly coercive formulation of Helmholtz),
- other boundary conditions (Dirichlet, Neumann),
- analysis of non-infsup-stable impedances,
- Maxwell, strongly elliptic problems.

Perspectives

- fine properties of the exchange operator, localization procedure
- large scale optimized parallel implementation (with CEA CESTA)
- multi-level strategy
- non-conforming DDM

Thank you for your attention

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E.Parolin, "Non-overlapping domain decomposition methods with non-local transmission operators for harmonic wave propagation problems", PhD thesis, IP Paris, 2020.

X.Claeys, "A new variant of the Optimised Schwarz Method for arbitrary non-overlapping subdomain partitions", ESAIM Math. Model. Numer. Anal. 55 (2021), no. 2, 429–448.

X.Claeys and E.Parolin, "Robust treatment of cross points in Optimized Schwarz Methods", Numer. Math. 151, No. 2, 405-442 (2022).

X.Claeys, "Non-self adjoint impedance in Generalized Optimized Schwarz Methods", submitted, preprint Arxiv 2108.03652

X.Claeys, F.Collino and E.Parolin, Matrix form of nonlocal OSM for electromagnetics, preprint Arxiv 2108.11352